Bachelorarbeit

A Transpose Channel Approach to Approximate Quantum Error Correction

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Introduction

Let us imagine a quantum computer. Its purpose is to perform operations on quantum information using typical quantum mechanical effects, i.e. superposition or entanglement. If we operate on quantum information we cannot prevent the quantum information from being affected by some kind of quantum noise such as decoherence. So, we want to achieve a quantum computation robust to quantum noise. At this point, the field of quantum error correction arises.
The purpose of this bachelor thesis is to give an approach to approximate quantum error correction conditions via the transpose channel as general recovery operation. After some mathematical basics we start our journey with the basics concepts of quantum error correction and give an example of a quantum code, called the *Shor code*, which is resistant do single qubit errors. Then, we directly continue by introducing the quantum error correction condition which gives us a powerful tool to check if a quantum code meets our specific needs. After introducing the transpose channel as general correction operation we then show that this specific operation can be used to generalize the perfect quantum error conditions to include approximately correcting codes. In particular, it will yield into the main result of this bachelor thesis, the approximate quantum error correction conditions (AQEC conditions - first introduced by Ng and Mandayam). Further, we will introduce a generalization of this conditions for non trace preserving errors. Equipped with this tools we then finish our journey with a particular example of an approximately correcting code, the π-cat state code.
Our map for this journey on approximate quantum error correction will mostly be the paper *A simple approach to approximate quantum error correction* by Hui Khoon Ng and Prabha Mandayam from the Institute for Quantum Information, California Institute of Technology, Pasadena which was submitted on 4th of September in 2009.

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1 Basics

Our mathematical analyses are always motivated by physical processes. Therefore we make some postulates.

(1.1) Definition (State space)
We postulate that any state of an isolated physical system is described by its *state vector*, commonly denoted by $|\psi\rangle$ which lies in a complex Hilbert space. The space of all possible state vectors the system can take is defined as the system's *state space* which we denote by Q. Q is a Hilbert space itself and separable, i.e. there exists an orthonormal base denoted by $|k\rangle_{k\in\mathbb{N}}$. ◇

In line with the first definition we will continue to use the bra-ket notation in this bachelor thesis. In this context, we distinguish one special class of states from the others.

(1.2) Definition (Pure state)
A *pure state* of a quantum system is denoted by a state vector $|\psi\rangle$ of unit length, i. e. $\langle\psi|\psi\rangle = 1$. ◇

Further, we do not want to consider all operators acting on Q but only special ones:

(1.3) Definition (Quantum operator)
A *quantum operator* is a trace class Hermitian operator acting between state spaces. The set of all quantum operators between two state spaces Q_1, Q_2 is denoted by $O(Q_1, Q_2)$. If $Q_1 = Q_2$ we also write $O(Q_1)$ instead of $O(Q_1, Q_1)$. ◇

One particular operator describes the state of the quantum system completely and will be used very frequently in the following. It is called the *density operator* and given by

(1.4) Definition (Density operator)
Let Q be the state space induced by the orthonormal base $|k\rangle_{k\in\mathbb{N}}$. Then, every possible quantum system is explicitly described by its density operator commonly denoted by $\rho \in O(Q)$ and defined by

$$\rho = \sum_k p_k |k\rangle \langle k| \quad \text{for particular } p_k \in \mathbb{R}^+ \text{ with } \sum_k p_k = 1.$$

The set of all density operators of Q is denoted by $\mathfrak{D}(Q)$. ◇

Instead of a *density* operator we will often use the term *density matrix* connoting the same meaning.
In order to describe the evolution of a quantum system which includes the variation caused by an error we use the quantum operations formalism.

(1.5) Definition (Quantum operation)
A *quantum operation* $\mathfrak{E}$ is a linear map between spaces of density operators of state spaces, i.e. $\mathfrak{E} : \mathfrak{D}(Q_1) \to \mathfrak{D}(Q_2)$, which satisfies the following three axioms:

1. For every density operator ρ of Q_1 it is $0 \leq tr(\mathfrak{E}(\rho))_{Q_2} \leq 1$
2. $\mathfrak{E}$ is *completely positive*, that is for every positive operator $A \in \mathfrak{D}(Q_1)$ the image $\mathfrak{E}(A) \in \mathfrak{D}(Q_2)$ is also positive. Furthermore, for every finite dimensional state space R the operator $I_R \otimes \mathfrak{E}$ is completely positive when I_R denotes the identity on R.
3. $\mathfrak{E}$ is a *convex-linear* map, that is

$$\mathfrak{E}(\sum_{i=1}^{n} p_i A_i) = \sum_{i=1}^{n} p_i \mathfrak{E}(A_i) \quad \text{for } n \in \mathbb{N} \text{ and } A_i \in \mathfrak{D}(Q_1) \text{ for all } i. \qquad \diamond$$

The three axioms originate from physical requirements. When dealing with measurements it is beneficial not to restrict $\mathfrak{E}$ as trace preserving. Instead, we take $tr(\mathfrak{E}(\rho))$ as the probability that the outcome described by $\mathfrak{E}$ occurs on measuring. This induces the first axiom while the second axiom ensures that $\mathfrak{E}$ applied on an ensemble of quantum states ($\rho = \sum_i p_i \rho_i$) acts as intended. At last, the second axiom stems from the requirement that not only $\mathfrak{E}(\rho)$ is a valid density matrix in its own closed quantum system but also when seen in context with the environment.
There is a strong connection between quantum operations and quantum operators.

(1.6) Theorem (Operator-sum representation)
Let $\mathfrak{E} : \mathfrak{D}(Q_1) \to \mathfrak{D}(Q_2)$ be a quantum operation for state spaces Q_1 and Q_2 of finite dimension n resp. m. Then, there exists a set of operators $\{E_k \in O(Q_1, Q_2)\}_{1 \leq k \leq nm}$ such that

$$\mathfrak{E}(\rho) = \sum_{k=1}^{nm} E_k \rho E_k^{\dagger}.$$

Conversely, any map $\mathfrak{E}$ with this property is a quantum operation if $\sum_{k=1}^{nm} E_k E_k^{\dagger} \leq I$, i. e., $I - \sum_{k=1}^{nm} E_k E_k^{\dagger}$ is positive.
The operators E_k are called *operation elements* or *Kraus operators*. $\diamond$

From now on we will only consider finite state spaces. Due to the fact that for every possible quantum system only a finite number of different orthogonal states is significant (i.e. the energy is always finite), this is a valid restriction.

(1.7) Theorem (Unitary freedom in the ensemble)
Let two ensembles $A := \{\sqrt{p_i}\,|\psi_i\rangle\, | 1 \leq i \leq n\}$ and $B := \left\{\sqrt{q_j}\,|\phi_j\rangle\, | 1 \leq j \leq m\right\}$ of pure states be given. We may ensure that $m = n$ after appending zero vectors to the shorter list. Then, it is

$$\rho_A = \sum_{i=1}^{n} p_i \,|\psi_i\rangle \langle\psi_i| = \sum_{j=1}^{n} q_j \,|\phi_j\rangle \langle\phi_j| = \rho_B$$

if and only if there exist an unitary matrix $u \in \mathbb{C}^{n\times n}$ such that

$$\sqrt{p_i}\,|\psi_i\rangle = \sum_{j=1}^{n} u_{ij} \sqrt{q_j}\,|\phi_j\rangle \quad \text{for all } 1 \leq i \leq n. \qquad \diamond$$

Proof
For reasons of legibility we define $\left|\psi_i'\right\rangle := \sqrt{p_i}\left|\psi_i\right\rangle$ and $\left|\phi_j'\right\rangle := \sqrt{q_j}\left|\phi_j\right\rangle$ for all $1 \leq i,j \leq n$.
We first proof the "if" by using directly the unitary property of u. For this, let $\left|\psi_i'\right\rangle = \sum_{j=1}^n u_{ij}\left|\phi_j'\right\rangle$ for all $1 \leq i \leq n$ and some unitary matrix $u \in \mathbb{C}^{n\times n}$. Then,

$$\begin{aligned}\rho(A) &= \sum_{i=1}^n p_i \left|\psi_i\right\rangle\left\langle\psi_i\right| = \sum_{i=1}^n \left|\psi_i'\right\rangle\left\langle\psi_i'\right| = \sum_{1\leq i,j,k\leq n} u_{ij}u_{ik}^* \left|\phi_j'\right\rangle\left\langle\phi_k'\right| \\ &= \sum_{1\leq j,k\leq n} \underbrace{\left(\sum_{i=1}^n u_{kj}^\dagger u_{ij}\right)}_{=\delta_{kj},\, u \text{ is unitary}} \left|\phi_j'\right\rangle\left\langle\phi_k'\right| = \sum_{j=1}^n \left|\phi_j'\right\rangle\left\langle\phi_j'\right| \\ &= \sum_{j=1}^n q_j\left|\phi_j\right\rangle\left\langle\phi_j\right| = \rho(B)\end{aligned}$$

so A and B generate the same density matrix.
On the other hand, let

$$\rho = \sum_{i=1}^n \left|\psi_i'\right\rangle\left\langle\psi_i'\right| = \sum_{j=1}^n \left|\phi_j'\right\rangle\left\langle\phi_j'\right|.$$

We will show that there exists an unitary matrix $u \in \mathbb{C}^{n\times n}$ such that $\left|\psi_i'\right\rangle = \sum_{j=1}^n u_{ij}\left|\phi_j'\right\rangle$ for all $1 \leq i \leq n$. Since ρ is Hermitian and the span of A (as well as of B) is a finite subset of a Hilbert space, we have a spectral decomposition. That is, there exists an orthonormal basis $\{\left|k\right\rangle | 1 \leq k \leq N\}$ with $N \leq n$ such that $\rho = \sum_{k=1}^N \alpha_k \left|k\right\rangle\left\langle k\right|$ for $\alpha_k \in \mathbb{R}^{>0}$. Then, we have an orthogonal basis defined by the $\left|k'\right\rangle = \sqrt{\alpha_k}\left|k\right\rangle$ such that

$$\rho = \sum_{k=1}^N \left|k'\right\rangle\left\langle k'\right|.$$

Further, each $\left|\psi_i'\right\rangle$ can be written in terms of $\left|k'\right\rangle$. Otherwise, one $\left|\psi_i'\right\rangle$ must have a component orthogonal to all $\left|k'\right\rangle$ denoted by $\left|K^\perp\right\rangle$. Then, $\left\langle K^\perp|\psi_i'\right\rangle \neq 0$ whereas

$$\left\langle K^\perp\right|\rho\left|K^\perp\right\rangle = \sum_{k=1}^N \underbrace{\left\langle K^\perp|k'\right\rangle}_{=0}\underbrace{\left\langle k'|K^\perp\right\rangle}_{=0} = 0 = \sum_{j=1}^n \left\langle K^\perp|\psi_i'\right\rangle\left\langle\psi_i'|K^\perp\right\rangle = \sum_{j=1}^n \left|\left\langle K^\perp|\psi_i'\right\rangle\right|^2$$

which implies $\left\langle K^\perp|\psi_i'\right\rangle = 0$ for all j; a contradiction. Thus, there exists a complex $n \times N$ matrix c such that

$$\left|\psi_i'\right\rangle = \sum_{k=1}^N c_{ik}\left|k'\right\rangle \quad \text{for all } 1 \leq i \leq n$$

and we get

$$\sum_{k=1}^{N} |k'\rangle \langle k'| = \sum_{i=1}^{n} |\psi_i'\rangle \langle \psi_i'| = \sum_{1\le k,l\le n} \left(\sum_{i=1}^{n} c_{ik} c_{il}^* \right) |k'\rangle \langle l'|$$

which implies (due to the linearly independency of the $|k'\rangle\,|l'\rangle$):

$$\text{For all } 1 \le k, l \le N : \left(c^\dagger c\right)_{kl} = \sum_{i=1}^{n} c_{ik} c_{il}^* = \delta_{kl}.$$

When appending extra columns to c if needed we may obtain an unitary matrix $v \in \mathbb{C}^{n\times n}$. After setting $|k'\rangle := 0$ for every $k > N$ we can conclude that there exists an unitary complex matrix v such that

$$|\psi_i'\rangle = \sum_{k=1}^{n} v_{ik} |k'\rangle \quad \text{for all } 1 \le i \le n.$$

Likewise, we may obtain another unitary matrix $w \in \mathbb{C}^{n\times n}$ such that

$$\left|\phi_j'\right\rangle = \sum_{k=1} w_{jk} |k'\rangle \quad \text{for all } 1 \le j \le n.$$

Now, we can combine those two results to get an unitary matrix $u := vw^\dagger$ such that

$$|\psi_i\rangle = \sum_{j=1}^{n} |\phi_j\rangle \quad \text{for all } 1 \le i \le n. \qquad \square$$

(1.8) Theorem (Unitary freedom in the operator-sum representation)
Let two quantum operations denoted by $\mathfrak{E}(\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger$ and $\mathfrak{F} = \sum_{j=1}^{m} F_j \rho F_j^\dagger$ for all operators ρ acting on a given state space Q be given. We may ensure that $m = n$ after appending zero operators to the sum. Then, $\mathfrak{E} = \mathfrak{F}$ if and only if there exist an unitary matrix $u \in \mathbb{C}^{n\times n}$ such that

$$E_i = \sum_{j=1}^{n} u_{ij} F_j \quad \text{for all } 1 \le i \le n \qquad \diamond$$

Proof
Let $E_i = \sum_{j=1}^{n} u_{ij} F_j$ for all $1 \le i \le n$. Then

$$\mathfrak{E}(\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger = \sum_{1\le j,j'\le n} \underbrace{\sum_{i=1}^{n} u_{ij} u_{ij'}^*}_{=\delta_{jj'},\, u \text{ is unitary}} F_j \rho F_{j'}^\dagger = \sum_{j=1}^{n} F_j \rho F_j^\dagger$$

for every $\rho \in Q$, i.e. $\mathfrak{E} = \mathfrak{F}$.
Conversely, let $\mathfrak{E}(\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger = \sum_{j=1}^{n} F_j \rho F_j^\dagger$ for all $\rho \in \mathfrak{D}(Q)$. We will mainly use (1.7) to proof the existence of an unitary matrix u which relates the two quantum operations as desired. However, applying (1.7) directly on $\mathfrak{E}(\rho)$ and $\mathfrak{F}(\rho)$ for some special ρ would not accomplish our goal. Instead, the trick is to first define a special quantum operator σ acting on the state space of $Q^2 := Q \otimes Q'$ and in a second step apply (1.7) on σ:
Let $\{|k\rangle \,|1 \leq k \leq q\}$ be an orthonormal base for Q. We denote the orthonormal base for $Q \otimes Q'$ by $\{|k\rangle \otimes |k'\rangle \,|1 \leq k, k' \leq q\}$. Then define

$$|\alpha\rangle := \sum_{k=1}^{q} |k\rangle \langle k'|$$

which is a maximally entangled state of the system $Q \otimes Q'$. For $|\alpha\rangle$ define σ as the application of $\mathfrak{E}$ to one half of this maximally entangled state, i.e.

$$\sigma := \left(\mathfrak{E} \otimes I_{Q'}\right) (|\alpha\rangle \langle\alpha|). \tag{1}$$

We note that for this σ we have

$$\begin{aligned}
\sigma &= \left(\mathfrak{E} \otimes I_{Q'}\right) \big(\sum_{1\leq k,l\leq q} |k\rangle \left|k'\right\rangle \langle l| \left\langle l'\right| \big) = \mathfrak{E}\left(\sum_{1\leq k,l\leq q} |k\rangle \langle l| \right) \otimes \left|k'\right\rangle \left\langle l'\right| \\
&= \sum_{i=1}^{n} \sum_{1\leq k,l\leq q} (E_i |k\rangle \langle l| E_i^\dagger) \otimes (\left|k'\right\rangle \left\langle l'\right|) = \sum_{i=1}^{n} \sum_{1\leq k,l\leq q} (E_i |k\rangle \otimes \left|k'\right\rangle)(\underbrace{\langle l| E_i^\dagger \left\langle l'\right|}_{=(E_i|l\rangle\otimes|l'\rangle)^\dagger}) \\
&= \sum_{i=1}^{n} |e_i\rangle \langle e_i|
\end{aligned}$$

for

$$|e_i\rangle := \sum_{k=1}^{q} E_i |k\rangle \otimes \left|k'\right\rangle \quad \text{for all } 1 \leq i \leq n$$

while E_i commutes with $|k'\rangle$ for all i, k due to the fact that E_i does not act on Q' but on Q.
Doing the same for $\mathfrak{F} = \mathfrak{E}$ and with $|f_i\rangle$ defined analogously to $|e_i\rangle$ yields

$$\sum_{i=1}^{n} |e_i\rangle \langle e_i| = \sigma = \sum_{i=1}^{n} \sum_{1\leq k,l\leq q} \left(F_i |k\rangle \otimes \left|k'\right\rangle\right) \left(F_i |l\rangle \otimes \left|l'\right\rangle\right)^\dagger = \sum_{i=1}^{n} |f_i\rangle \langle f_i|.$$

Since this is the exact condition of (1.7), there exists an unitary matrix $u \in \mathbb{C}^{n\times n}$ such that

$$|e_i\rangle = \sum_{j=1}^{n} u_{ij} \left|f_j\right\rangle.$$

However, this means for every $|k\rangle$ and arbitrary i

$$E_i |k\rangle = E_i |k\rangle \otimes \langle k'|k'\rangle = \langle k'|e_i\rangle = \sum_{j=1}^{n} u_{ij} \langle k'|f_j\rangle = \sum_{j=1}^{n} u_{ij} F_j |k\rangle$$

and so

$$E_i = \sum_{j=1}^{n} u_{ij} F_j \quad \text{for all } 1 \leq i \leq n. \qquad \square$$

2 Theory of Quantum Error Correction

In this chapter we want to examine the effects of quantum noise on quantum codes. In particular, we will discuss some common codes and concepts of error correction. The final result will be a characterization on the correctability of quantum errors and the conditions the code must fulfill to correct some space of errors.

2.1 Basic Concepts of Error Correction

Whenever digital information is handled, errors may occur. To deal with this occurrence, we do not treat the information directly but we embed it into a *code*. Basically this means that we add enough other redundant information with the intention to be still able to deduce our actual information from the code after an error affected it. Formally:

(2.1) Definition (Code)
A code is an injective map between two finite sets. ◇

A simple example of a code which we all use every day is the repetition of the critical information several times, several cakes, several times. When taking a majority vote, we get that I meant *several times* in the last sentence. So, the cake is a lie.
In times of digital information we deal with bits instead of whole phrases. For this, we assume that the errors are independent and act with some probability p on one bit. Errors not matching these conditions will not be considered. This is valid since it is less likely that an error occurs acting on more than one bit. Furthermore, we want to concentrate on quantum processes inducing quantum information. So, we have to deal with the quantum environment which differs from the classical assumptions. First of all, we introduce the quantum analogue of the classical bit: the *qubit* or quantum bit. It is defined as the unit of quantum information. While a qubit can also have two possible values 0 or 1 it does not have to choose between the two values but can be a superposition of both -in spite of a classical bit which must be either 0 or 1. We still want to consider single qubit errors only. The typical error affecting only one qubit is the *bit-flip error*, i.e. it changes 0 to 1 and 1 to 0. It can be represented by the Pauli X operator. The Pauli operators are very useful when examining errors and their corrections. They are given by

(2.2) Definition (Pauli operators)
The Pauli matrices $X, Y, Z \in \mathbb{C}^{2\times 2}$ denoted by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = iXZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

together with the identity, span the full complex vector space of 4-dimensional Hermitian matrices. ◇

An operator X that onky acts on the i-th qubit of a code $|a_1 a_2 \dots a_i \dots a_n\rangle$ is indicated by X_i. Since every error acting on one qubit can be interpreted as an 2×2 Hermitian complex matrix and Y is a combination of X and Z itself, we have

(2.3) Remark
Every single qubit error can be written as a linear combination of the Pauli matrices X and Z. ◇

The challenge in quantum error correction lies in the advantage of an information system based on quantum processes: We cannot simply measure an affected encoded state to detect the error since this would force its immediate collapse and therefore would destroy the information stored in the superposition.

2.2 Shor's Code

Due to the *no-cloning theorem* which shows that a qubit cannot be copied, it is not possible to implement our repetition code example into a quantum process. However, it is possible to spread the information of one qubit onto several qubits which are highly-entangled. This method was first discovered by *Peter Shor* and is called the *Shor's code*:

(2.4) Definition (Shor's Code)
The *Shor's code* is defined by the map

$$\begin{aligned} |0\rangle &\mapsto |\bar{0}\rangle := (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ |1\rangle &\mapsto |\bar{1}\rangle := (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \end{aligned}$$ ◇

We want to examine the code's ability in error correction. For this, suppose the encoded state $|\bar{\psi}\rangle$ is affected by a qubit flip error on the i-th qubit. To detect the error we apply the same procedure to every block of three qubits starting with the first one:
At the beginning, we compare the first and second qubit of the current block via the parity check $Z_1 Z_2$ and get the outcome $r_1 \in \{-1, +1\}$. That is, we check whether both qubits have the same value. Then, we apply the same procedure to the second and third qubit, i.e. measuring $Z_2 Z_3$ and getting $r_2 \in \{-1, +1\}$. The combination of r_1 and r_2 then explicitly yields the kind of qubit flip error:

1. $(r_1, r_2) = (1, 1) \rightarrow$ no error
2. $(r_1, r_2) = (-1, 1) \rightarrow$ qubit flip on the first qubit, i.e. X_1
3. $(r_1, r_2) = (-1, -1) \rightarrow$ qubit flip on the third qubit, i.e. X_2
4. $(r_1, r_2) = (1, -1) \rightarrow$ qubit flip on the third qubit, i.e. X_3

Applying this to every block of three qubits detects any single qubit Pauli error, which then can be corrected by applying X_i (for particular i). Indeed, Shor's 9-qubit code was the first quantum error-correcting code encoding a single qubit and correcting any single qubit Pauli error.

2.3 Quantum Error Correction Conditions

The basic concept behind every quantum code (including Shor's code we introduced before) is to embed the state space Q into some larger code space C with the aim of gaining the ability to distinguish the sets of noised states $\mathfrak{E}(\rho)$ from each other. This means, we want to be able to do a *syndrome* measurement on the erred encoded state to identify the type of error, the *error-syndrome*, that occurred. Then, having identified the error, the *recovery* can be applied to gain the quantum system of the original state of the quantum code. This bundle of error-detection and recovery builds the error-correction operation denoted by $\mathfrak{R}$. It has to be trace preserving since it must succeed with probability 1. The other requirement for $\mathfrak{R}$ to be considered successful is

$$\mathfrak{R} \circ \mathfrak{E}(\rho) \propto \rho \quad \text{for all } \rho \text{ whose support lies in } C. \tag{2}$$

In the following, we will show that if a set of errors is correctable it is possible to construct an error-correction operation built of two parts: error-detection and recovery. However, we first need to find a practical way to determine the correctability of a set of errors. We consider the special case that different error syndromes correspond to different orthogonal subspaces. Due to the orthogonality of the subspaces we can easily discern the errors. With the knowledge of the particular error syndrome that affected the qubit we then can perform a specific recovery. Indeed, we have the following characterization of a correctable set of errors:

(2.5) Theorem (Quantum error-correction conditions (QEC), [5])
Let P be the projector onto a quantum code $\mathfrak{C}$ of dimension n. A given set of errors $\{E_i | 1 \leq i \leq m\}$ is correctable if and only if

$$P\, E_i^\dagger E_j P = \alpha_{ij} P \quad \text{for all } 1 \leq i, j \leq m \tag{3}$$

for some complex matrix $\alpha \in \mathbb{C}^{m \times m}$. ◇

Proof
Let (3) be satisfied for $E = \{E_i \,| i = 1, \dots, k\}$ inducing the quantum operation $\mathfrak{E}(\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger$. We will proof that E is correctable by constructing a two-part error-correction operation $\mathfrak{R}$. "Two-part" implies: error-detection first and then recovery,

i.e. the same form of procedure we used before.
Due to

$$\alpha_{ij}^* P = (\alpha_{ij} P)^\dagger = (P\, E_i^\dagger E_j P)^\dagger = P\, E_j^\dagger E_i P = \alpha_{ji} P \quad \text{for every } 1 \le i,j \le k$$

α is a Hermitian matrix. Therefore, it can be transformed with an unitary matrix $u \in \mathbb{C}^{m\times m}$ into a diagonal matrix $d \in \mathbb{C}^{m\times m}$ via $d = u^\dagger \alpha u$. The set of $F_k := \sum_{i=1}^m u_{ik} E_i$ with $1 \le k \le m$ satisfies the conditions of (1.8) since u is unitary. Thus, $\mathfrak{E} = \sum_{i=1}^m E_i \rho E_i^\dagger = \sum_{j=1}^m F_j \rho F_j^\dagger$ such that $F := \{F_k | \, 1 \le k \le m\}$ is also a set of operation elements for $\mathfrak{E}$. Furthermore, it is

$$P\, F_k^\dagger F_l P = \sum_{1\le i,j\le m} P\, u_{ki}^\dagger E_i u_{lj} E_j P = \sum_{1\le i,j\le m} u_{ki}^\dagger u_{lj} P\, E_i E_j P \underbrace{=}_{(3)} \underbrace{\sum_{1\le i,j\le m} u_{ki}^\dagger u_{lj} \alpha_{ij}}_{=d_{kl}} P = d_{kl} P$$

for all $1 \le k \le m$. So, F even fulfills the error-correction condition (3) for the diagonal Hermitian matrix d. The polar decomposition gives us the existence of an unitary $U_k \in \mathfrak{D}(Q)$ such that

$$F_k P = U_k \sqrt{P\, F_k^\dagger F_k P} = \sqrt{d_{kk}} U_k P \tag{4}$$

for all $1 \le k \le n$. This shows us how F_k actually effects the code space since the Hermitian matrix indicates a rotation. In fact, F_k rotates the code space defined by P into the subspace defined by $P_k := U_k P U_k^\dagger = F_k / d_{kk}\, P F_k^\dagger$. These subspaces are orthogonal since

$$P_l P_k^\dagger = \frac{U_l \overbrace{P F_l^\dagger F_k P}^{=\delta_{kl} d_{kk} P} U_k^\dagger}{\sqrt{d_{kk} d_{ll}}} = \delta_{kl} U_l P U_k^\dagger$$

for all $1 \le l,k \le n$. After adding $P_{n+1} := I - \sum_{k=1}^n P_k$, we may trivially ensure that this set of projectors fulfills the completeness relation, i.e. $\sum_{k=1}^{n+1} P_k = I$, such that these projectors define a syndrome measurement; the first part of the error-correction operation. Since there exists no F_k which rotates the code space into the subspace defined by P_{n+1} due to its definition, P_{n+1} is orthogonal to $im(\mathfrak{E})$ and therefore can be suppressed in the following.
When knowing the error (which means knowing the particular k) the second part, the recovery, then is done by just applying $U_k^\dagger$. The combination of these two steps is

$$\mathfrak{R}(\sigma) = \sum_{k=1}^n U_k^\dagger P_k \sigma P_k U_k \tag{5}$$

so $\mathfrak{R} \sim \left\{ P U_k^\dagger \right\}$. Since P is a projection it acts on C as identity and $P^2 = P$ such that

$$U_k^\dagger P_k F_l \rho = U_k^\dagger P_k F_l P \rho = \underbrace{U_k^\dagger U_k}_{=I} \overbrace{P U_k^\dagger}^{=(U_k P)^\dagger = (F_k/\sqrt{d_{kk}} P)^\dagger} F_l P \rho \; = \frac{P F_k^\dagger F_l P \rho}{\sqrt{d_{kk}}} = \delta_{kl} \sqrt{d_{kk}} P \rho = \delta_{kl} \sqrt{d_{kk}} \rho \tag{6}$$

for all $1 \leq k, l \leq n$. However, this yields

$$\begin{aligned}
\mathfrak{R}(\mathfrak{E}(\rho)) &= \sum_{1 \leq k,l \leq n} U_k^\dagger P_k F_l \rho F_l^\dagger P_k U_k = \sum_{k=1}^{n} \rho F_k^\dagger P_k U_k \\
&= \sum_{k=1}^{n} \sqrt{d_{kk}} (U_k^\dagger P_k^\dagger F_k \rho^\dagger)^\dagger = \sum_{k=1}^{n} \sqrt{d_{kk}} (U_k^\dagger P_k F_k \rho)^\dagger \\
&= \sum_{k=1} d_{kk} \rho = tr(d) \rho \propto \rho
\end{aligned}$$

so $\mathfrak{R}$ corrects $\mathfrak{E}$ as desired.
Conversely, assume $\{E_i | 1 \leq i \leq m\}$ is a set of errors which is correctable by the error-correction operation $\mathfrak{R}(\sigma) := \sum_{k=1}^{n} R_k \sigma R_k^\dagger$. Since $P\rho P$ lies in C we have

$$\begin{aligned}
& \mathfrak{R} \circ \mathfrak{E}(P\rho P) \propto P\rho P \Leftrightarrow \mathfrak{R} \circ \mathfrak{E}(P\rho P) = cP\rho P \quad \text{for } c \in \mathbb{C} \text{ and all } \rho \in O(Q) \\
\Leftrightarrow \quad & \textstyle\sum_{k=1}^{m} \sum_{i=1}^{n} R_k E_i P \rho P E_i^\dagger R_k^\dagger = cP\rho P \quad \text{for } c \in \mathbb{C} \text{ and all } \rho \in O(Q)
\end{aligned}$$

so the quantum operation with operation elements $\{R_k E_i P | 1 \leq k \leq m, 1 \leq i \leq n\}$ is the same as the one with operation elements $\left\{\sqrt{c}P\right\}$. Applying (1.8) gives the existence of $c_{ki} \in \mathbb{C}$ for all $1 \leq k \leq m$ and $1 \leq i \leq n$ such that

$$R_k E_i P = c_{ki} P \text{ as well as } P E_i^\dagger R_k^\dagger = c_{ki}^* P.$$

Due to the fact that by definition $\mathfrak{R}$ is a trace preserving operation, i.e. $\sum_{k=1}^{m} R_k^\dagger R_k = I$, it is

$$\begin{aligned}
\sum_{k=1}^{m} P E_i^\dagger R_k^\dagger R_k E_j P &= P E_i^\dagger E_j P \\
&= \underbrace{\sum_{k=1}^{n} c_{ki}^* c_{kj}}_{:=\alpha_{ij}} P
\end{aligned}$$

for all $1 \leq i, j \leq n$. Further, it is obvious that $\alpha \in \mathbb{C}^{n \times n}$ is Hermitian such that all quantum error-correction conditions are fulfilled. □

As shown in the proof it is useful to choose a special Kraus representation of $\mathfrak{E}$ which induces a diagonal form of the QEC.

(2.6) Corollary
Let P be the projector onto a quantum code $\mathfrak{C}$ of dimension n. An error $\mathfrak{E}$ is correctable if and only if there exists a Kraus representation $\{F_k | 1 \leq k \leq m\}$ of $\mathfrak{E}$ such that

$$P F_k^\dagger F_l P = d_{kk} P \quad \text{for all } 1 \leq k, l \leq m \tag{7}$$

for some diagonal matrix $d \in \mathbb{C}^{m \times m}$. ◇

(2.7) Corollary
Let $\{E_i|1 \leq i \leq n\}$ be a correctable set of errors that induce the noise process $\mathfrak{E}$ and $\mathfrak{R}$ an error-correction operation correcting $\mathfrak{E}$. Then, $\mathfrak{R}$ corrects every other quantum operation $\mathfrak{F}$ which operation elements $\{F_j|1 \leq j \leq m\}$ are linear combinations of the operation elements of $\mathfrak{E}$, i.e. $F_j = \sum_{i=1}^{n} c_{ji} E_i$ for every j and for some complex matrix $c \in \mathbb{C}^{m\times n}$. ◇

Proof
We deduce this claim from the error-correction conditions by picking up some particular results of its proof.
From (2.5) follows that the operation elements $\{E_i\}$ must fulfill the quantum error-correction conditions

$$PE_iE_j^\dagger P = d_{ij}P$$

As shown in the proof of (2.5) we may assume that d is diagonal without loss of generality. We use the same notation as in the proof so $\mathfrak{R}$ has operation elements $U_k^\dagger P_k$ and (6) holds:

$$U_k^\dagger P_k E_i \rho = \delta_{ki}\sqrt{d_{kk}}\rho$$

Substituting $F_j = \sum_{i=1}^{n} c_{ji} E_i$ yields

$$\begin{aligned} U_k^\dagger P_k F_j \rho &= \sum_{i=1}^{n} c_{ji} U_k^\dagger P_k E_i \rho = \sum_{i=1}^{n} c_{ji}\delta_{ki}\sqrt{d_{kk}}\rho \\ &= c_{jk}\sqrt{d_{kk}}\rho \end{aligned}$$

such that

$$\begin{aligned} \mathfrak{R}(\mathfrak{F}(\rho)) &= \sum_{k=1}^{n}\sum_{j=1}^{m} U_k^\dagger P_k F_j \rho F_j^\dagger P_k U_k \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n} \left|c_{jk}\right|^2 d_{kk}\rho \propto \rho. \end{aligned}$$

So, $\mathfrak{R}$ corrects $\mathfrak{F}$. □

The recovery map $\mathfrak{R}$ constructed in the previous proof can be identified with a map called *transpose channel* first defined in [2]. For this, we upgrade the previous corollary (2.6) to:

(2.8) Lemma (QEC feat. transpose channel)
Let P be the projector onto a quantum code $\mathfrak{C}$ of dimension n affected by an error $\mathfrak{E} \sim \{F_k|1 \leq k \leq m\}$. Let $P_{\mathfrak{E}}$ denote the projector onto $\mathfrak{P}_{\mathfrak{E}} := supp\,[\mathfrak{E}(\mathfrak{C})] = supp\,[\mathfrak{E}(P)]$. Then, the error $\mathfrak{E}$ is correctable and the recovery is given by the transpose channel $R_p : O(\mathfrak{P}_{\mathfrak{E}}) \to O(\mathfrak{C})$, termed

$$R_p(\cdot) = \sum_{i=1}^{n} PF_i^\dagger \mathfrak{E}(P)^{-1/2}(\cdot)\mathfrak{E}(P)^{-1/2}F_iP \qquad (8)$$

if and only if

$$PF_k^\dagger F_l P = d_{kk} P \quad \text{for all } 1 \leq k, l \leq m \tag{9}$$

for some diagonal matrix $d \in \mathbb{C}^{m\times m}$. ◇

Note, that the inverse of $\mathfrak{E}(P)$ is taken on its support $\mathfrak{P}_{\mathfrak{E}}$.

Proof
It is only left to show that $\mathfrak{R}_p$ as defined above corrects the error $\mathfrak{E}$. To do so we proof the equality of $\mathfrak{R}_p$ with the recovery map constructed in the proof of (2.5):
As already shown in (4) the polar decomposition yields $\mathfrak{E}(P) = \sum_{k=1}^m (F_k P)(PF_k^\dagger) = \sum_{k=1}^m d_{kk} P_k$ for orthogonal $P_k = U_k P U_k^\dagger$. Hence, it is $\mathfrak{E}(P)^{-1/2} = \sum_{k=1}^m P_k/\sqrt{d_{kk}}$. Due to this equation, the Kraus operators $\{PF_k^\dagger \mathfrak{E}(P)^{-1/2}\}$ of the transpose channel $\mathfrak{R}_p$ can be written as

$$\begin{aligned} PF_k^\dagger \mathfrak{E}(P)^{-1/2} &= \sum_{l=1}^m \underbrace{\sqrt{d_{kk}} P U_k^\dagger}_{\text{polar decomp.}} \frac{P_l}{\sqrt{d_{ll}}} = \sqrt{d_{kk}} U_k^\dagger \sum_{l=1}^m \underbrace{U_k P U_k^\dagger}_{=P_k} \frac{P_l}{\sqrt{d_{ll}}} \\ &= U_k^\dagger P_k^2 = U_k^\dagger P_k = P U_k^\dagger. \end{aligned} \tag{10}$$

Now, with (5) we see that the Kraus representations are equal and therefore the operators have to be equal, too. □

We note that the transpose channel's domain is $O(P_{\mathfrak{E}}) \subset O(\mathfrak{H})$. Furthermore, since the transpose channel satisfies

$$\sum_k (PE_k^\dagger \mathfrak{E}(P)^{-1/2})^\dagger (PE_k^\dagger \mathfrak{E}(P)^{-1/2}) = \mathfrak{E}(P)^{-1/2} \sum_k E_k P E_k^\dagger \, \mathfrak{E}(P)^{-1/2} = P_{\mathfrak{E}} \tag{11}$$

it is trace preserving on its domain. We can extend $\mathfrak{R}_p$ to enforce it to be trace preserving on the whole space $\mathfrak{H}$. We just have to add an additional projector to the Kraus operators of $\mathfrak{R}_p$ that is the identity on the complement of $\mathfrak{P}_{\mathfrak{E}}$, i.e. $1 - P_{\mathfrak{E}}$. However, we assume that all information is stored in the code space $\mathfrak{C}$ so that it is irrelevant for us how $\mathfrak{R}_p$ acts outside of $O(P_{\mathfrak{E}})$. Thus, we neglect this extension.
Concluding, this lemma (2.8) leads to an alternate form of the QEC conditions (2.5):

(2.9) Theorem
Let P be the projector onto a quantum code $\mathfrak{C}$ of dimension n. A given set of errors $\{E_i | 1 \leq i \leq m\}$ is correctable if and only if

$$PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P = \beta_{ij} P \quad \text{for all } 1 \leq i, j \leq m \tag{12}$$

where $\beta := \sqrt{\alpha}$ with α as defined in equation (3). ◇

Proof
Let $\mathfrak{E}$ be correctable, i.e. the QEC conditions are fulfilled (since it is equivalent, we take the diagonal form of QEC conditions). Using equation (10) we get

$$PF_k^\dagger \mathfrak{E}(P)^{-1/2} F_l P = PU_k^\dagger F_l P \overset{\text{eq. (4)}}{=} \delta_{kl} \sqrt{d_{kk}} P \tag{13}$$

where U_k is induced by the polar decomposition of $F_k = \sum_{i=1}^{m} u_{ik} E_i$ with an unitary matrix u fulfilling $\alpha = u D u^\dagger$ for a diagonal matrix D. Thus, the code satisfies equation (12) with $\beta := \sqrt{\alpha}$.
On the other hand, let (12) be true. By choosing u so that α is diagonal (see (2.5) we can take β to be diagonal with entries $\sqrt{d_{kk}}$. Thus, we have the same equation (13) as before. Now, with $X := \mathfrak{E}(P)^{-1/4} F_k P$ we have $X^\dagger X = \sqrt{d_{kk}} P$ due to equation (13) and therefore it is $X = d_{kk}^{1/4} V_k P$ or equivalent $F_k P = d_{kk}^{1/4} \mathfrak{E}(P)^{1/4} V_k P$ for some unitary V_k. Using this on equation (13) yields $P V_k^\dagger V_l P = \delta_{kl} P$. On the other hand, it is

$$\mathfrak{E}(p) = \sum_k F_k P\, P F_k^\dagger = \sum_k \mathfrak{E}(P)^{1/4} \sqrt{d_{kk}} V_k P V_k^\dagger \mathfrak{E}(P)^{1/4}$$

such that $\mathfrak{E}(P)^{1/2} = \mathfrak{E}(P)^{-1/4} \mathfrak{E}(P) \mathfrak{E}(P)^{-1/4} = \sum_k \sqrt{d_{kk}} V_k P V_k^\dagger$.Combining these last results yields

$$\begin{aligned} P F_K^\dagger F_l P &= d_{kk}^{1/4} P V_k^\dagger\, \mathfrak{E}(P)^{1/4} \mathfrak{E}(P)^{1/4}\, V_l P d_{ll}^{1/4} = \sum_m d_{kk}^{1/4} d_{ll}^{1/4} \sqrt{d_{mm}}\, P \underbrace{V_k^\dagger V_m}_{=\delta_{km}} P \underbrace{V_m^\dagger V_l}_{=\delta_{ml}} P \\ &= \delta_{kl} d_{kk} P \end{aligned}$$

which is the desired equation (3) of QEC conditions in diagonal form. □

There is a strong connection between the last theorem (2.9) and lemma (2.8) due to the fact that the left-hand side of equation (12) is a Kraus operator of $\mathfrak{R}_p \circ \mathfrak{E}$ since:

$$\sum_{i,j} (P E_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P)(P E_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P)^\dagger = \sum_i P E_i^\dagger \mathfrak{E}(P)^{-1/2} \mathfrak{E}(P) \mathfrak{E}(P)^{-1/2} E_i P. \quad (14)$$

As lemma (2.8) shows, the transpose channel $\mathfrak{R}_p$ of a code $\mathfrak{C}$ is capable of correcting any error $\mathfrak{E}$ fulfilling the QEC conditions (2.5). Thus, the last theorem's claim (2.9) that the code $\mathfrak{C}$ is correctable if and only if $\mathfrak{R}_p \circ \mathfrak{E} \propto P$, i.e. $\mathfrak{R}_p$ corrects the error $\mathfrak{E}$, is not surprising after all.

3 Approximate Quantum Error Correction

3.1 Optimizing Problem

Up to now, we limited our interest on QEC where the recovery map *perfectly* corrects the error. However, it will be useful to allow QEC where the error is not perfectly corrected but only approximately: the *approximate quantum error correction*. To work with approximate QEC (AQEC) we need to be able to evaluate how well an AQEC code protects the information. This can be quantified by the *fidelity* between the input qubit state and the resulting state after noise and recovery operation.

(3.1) Definition (Fidelity)
For any two states with corresponding density matrices ρ, σ the fidelity $F(\rho, \sigma)$ between these two states is given by

$$F(\rho, \sigma) = tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \qquad \diamond$$

It is not self-evident that this is a useful tool to determine how close two states are. However, it does well as a distance measurement as shown in ([5, pp. 409-416]. Some important properties are:

(3.2) Lemma (Fidelity Properties)

1. For a pure state $\rho = |\phi\rangle\langle\phi|$ and a quantum operation σ the fidelity is given by $F(|\phi\rangle, \sigma) := F(|\phi\rangle\langle\phi|, \sigma) = \sqrt{\langle\phi|\sigma|\phi\rangle}$
2. For a pure state $\rho = |\phi\rangle\langle\phi|$ and quantum operation A with Kraus representation $\{A_k | 1 \leq k \leq K\}$ it is $F[|\phi\rangle, A(\rho)] = \sqrt{\sum_{k=1}^{K} |\langle\phi| A_k |\phi\rangle|^2}$
3. $F(\rho, \sigma) = F(\sigma, \rho)$ for all ρ, σ, i.e. the fidelity is symmetric in its arguments
4. $F(\rho, \sigma) \in [0, 1]$ for all ρ, σ
5. $F = 0$ if and only if $supp(\rho)$ is orthogonal to $supp(\sigma)$
6. $F = 1$ if and only if $\rho = \sigma$
7. F is concave in both arguments $\diamond$

Proof
We will only proof the first two properties. For a proof of the others, see ([5, pp. 409-416].
1. Given a pure state $\rho = |\phi\rangle\langle\phi|$ it is $\rho^{1/2} = \rho = |\phi\rangle\langle\phi|$ and therefore

$$F(\rho, \sigma) = tr\sqrt{\rho\sigma\rho} = tr\sqrt{|\phi\rangle\langle\phi|\sigma|\phi\rangle\langle\phi|} = \sqrt{\langle\phi|\sigma|\phi\rangle}tr\sqrt{|\phi\rangle\langle\phi|} = \sqrt{\langle\phi|\sigma|\phi\rangle}.$$

2. This property also follows directly when computing:

$$\langle\phi| A(\rho) |\phi\rangle = \sum_{k=1}^{K} \langle\phi| \left(A_k |\phi\rangle \langle\phi| A_k^\dagger\right) |\phi\rangle = \sum_{k=1}^{K} |\langle\phi| A_k |\phi\rangle|^2$$

such that

$$F[|\phi\rangle, A(\rho)] = \sqrt{\langle\phi| A(\rho) |\phi\rangle} = \sqrt{\sum_{k=1}^{K} |\langle\phi| A_k |\phi\rangle|^2}. \qquad \square$$

Since we will mainly consider fidelity for a state before and after it is affected by a map Φ, we use the shorthand

$$F(|\phi\rangle, \Phi) := F[|\phi\rangle, \Phi(|\phi\rangle\langle\phi|)]. \tag{15}$$

Now, based on this fidelity measure we determine the effectiveness of a code $\mathfrak{C}$ at protecting the information of a state space Q against an error $\mathfrak{E}$ on how close the *worst-case fidelity* $\min_{\rho\in Q} F(\rho, P^{-1} \circ \mathfrak{R} \circ \mathfrak{E} \circ P)$ is to 1. Due to the concavity in both arguments of the fidelity, it is sufficient to minimize only over pure states in Q. The search for the best AQEC to protect against a given error $\mathfrak{E}$ therefore means to maximize the worst-case fidelity over P and $\mathfrak{R}$. This problem is termed by

$$\max_{P} \max_{\mathfrak{R}} \min_{|\phi\rangle\in Q} F(|\phi\rangle, P^{-1} \circ \mathfrak{R} \circ \mathfrak{E} \circ P) \tag{16}$$

while reaching the maximum worst-case fidelity 1 for a specific P and $\mathfrak{R}$ means we have perfect QEC.
For a given code $\mathfrak{C} \subset \mathfrak{H}$ this can be written as

$$\max_{\mathfrak{R}} \min_{|\phi\rangle\in\mathfrak{C}} F(|\phi\rangle, \mathfrak{R} \circ \mathfrak{E}) \tag{17}$$

where the recovery $\mathfrak{R}$ with the largest worst-case fidelity is denoted by $\mathfrak{R}_{op}$ and called the *optimal recovery*. From this we derive

(3.3) Definition (Fidelity Loss)
For a given code $\mathfrak{C}$ and recovery $\mathfrak{R}$ the *fidelity loss*, denoted $\eta_{\mathfrak{R}}$, is given by

$$\eta_{\mathfrak{R}} := 1 - \min_{|\phi\rangle\in\mathfrak{C}} F^2(|\phi\rangle, \mathfrak{R} \circ \mathfrak{E})$$

where $F^2(\cdot,\cdot) := [F(\cdot,\cdot)]^2$. $\diamond$

The fidelity loss for the optimal recovery $\mathfrak{R}_{op}$, denoted η_{op}, is given by $\eta_{op} = \min_{\mathfrak{R}} \eta_{\mathfrak{R}}$ and called the *optimal fidelity loss*. If $\eta_{op} \leq \epsilon$ for some $\epsilon \in [0,1]$, the corresponding code is said to be *ϵ-correctable*. For $\epsilon \ll 1$ the code is said to be approximately correctable.

3.2 Transpose Channel and AQEC

For a given error $\mathfrak{E}$ the problem is to find the best code $\mathfrak{C}$ and recovery $\mathfrak{R}_{op}$ to minimize the fidelity loss to η_{op}. We do not want to focus on finding the perfect code but rather on finding conditions for the existence of ϵ-correctable recovery maps. In the case of perfect QEC we have the transpose channel $\mathfrak{R}_p$ as recovery map. In the case of AQEC, however, $\mathfrak{R}_p$ does not need to be the optimal recovery $\mathfrak{R}_{op}$. As we will show in the following, it is not much worse than $\mathfrak{R}_{op}$ after all.

(3.4) Theorem
Let $\mathfrak{C}$ be a n-dimensional code with optimal fidelity loss η_{op} affected by an error $\mathfrak{E}$. For any pure state $|\phi\rangle \in \mathfrak{C}$ it is for the transpose channel $\mathfrak{R}_p$

$$F^2\left(|\phi\rangle, R_{op} \circ \mathfrak{E}\right) \leq \sqrt{1+(n-1)\eta_{op}} \cdot F\left(|\phi\rangle, R_p \circ \mathfrak{E}\right) \tag{18}$$

◇

Proof
Let $\mathfrak{E} \sim \{E_k\}$ and $\mathfrak{R}_{op} \sim \{R_l\}$. With lemma (3.2.2) we know that for any $|\phi\rangle$ the fidelity $F^2(|\phi\rangle, \mathfrak{R}_{op} \circ \mathfrak{E})$ is given by

$$\begin{aligned} F^2\left[|\phi\rangle, \mathfrak{R}_{op} \circ \mathfrak{E}(P_\phi)\right] &= \sum_{kl} |\langle\phi| R_k E_l |\phi\rangle|^2 \\ &= \sum_{kl} \left|\langle\phi| X_k^\dagger Y_l |\phi\rangle\right|^2 = \sum_{kl} |\langle X_k \phi | Y_l \phi\rangle|^2 \end{aligned}$$

if we define X_k and Y_l as follows

$$X_k^\dagger := R_k \mathfrak{E}(P)^{1/4}, \quad Y_l := \mathfrak{E}(P)^{-1/4} E_l. \tag{19}$$

Further, by using the Cauchy-Schwarz inequality two times we get

$$\begin{aligned} F^2\left[|\phi\rangle, \mathfrak{R}_{op} \circ \mathfrak{E}(P_\phi)\right] &= \sum_{kl} |\langle X_k\phi | Y_l\phi\rangle|^2 \leq \sum_{kl} \langle\phi| X_k^\dagger X_k |\phi\rangle \langle\phi| Y_l^\dagger Y_l |\phi\rangle \\ &\leq \sqrt{\sum_k \left|\langle\phi| X_k^\dagger X_k |\phi\rangle\right|^2 \sum_l \left|\langle\phi| Y_l^\dagger Y_l |\phi\rangle\right|^2} \\ &= \sqrt{\sum_k \left|\langle\phi| R_k \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle\right|^2 \sum_l \left|\langle\phi| E_l^\dagger \mathfrak{E}(P)^{-1/2} E_l |\phi\rangle\right|^2}. \end{aligned}$$

Next, we will find useful upper bounds for each sum starting with the second sum, i.e. $\sum_l \left|\langle\phi| E_l^\dagger \mathfrak{E}(P)^{-1/2} E_l |\phi\rangle\right|^2$.
By adding only positive terms and due to the fact that P acts as identiy on $|\phi\rangle \in \mathfrak{C}$ we get

$$\sum_l \left|\langle\phi| E_l^\dagger \mathfrak{E}(P)^{-1/2} E_l |\phi\rangle\right|^2 \leq \sum_{kl} \left|\langle\phi| E_k^\dagger \mathfrak{E}(P)^{-1/2} E_l |\phi\rangle\right|^2 = \sum_{kl} \left|\langle\phi| P E_k^\dagger \mathfrak{E}(P)^{-1/2} E_l P |\phi\rangle\right|^2$$

such that the operator in the bracket becomes a Kraus operator for $\mathfrak{R}_p \circ \mathfrak{E}$ as shown in (14). Using (3.2.2) again then yields

$$\sum_l \left| \langle\phi| E_l^\dagger \mathfrak{E}(P)^{-1/2} E_l |\phi\rangle \right|^2 \leq \sum_{kl} \left| \langle\phi| P E_k^\dagger \mathfrak{E}(P)^{-1/2} E_l P |\phi\rangle \right|^2 = F^2 \left(|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E} \right). \quad (20)$$

We proceed with the first sum, i.e.

$$\sum_k \left| \langle\phi| R_k \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle \right|^2 = \sum_k \langle\phi| R_k \mathfrak{E}(P)^{1/2} R_k^\dagger P_\phi R_k \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle .$$

Since $\mathfrak{C}$ is a n-dimensional subspace we are free to choose a basis $\{|\phi_i\rangle\}_{i=1}^n$ such that $|\phi_1\rangle = |\phi\rangle$. Therefore, we can expand P_ϕ to P via $P = P_\phi + \sum_{i=2}^n |\phi_i\rangle \langle\phi_i|$. This yields

$$\begin{aligned} \sum_k \langle\phi| R_k \mathfrak{E}(P)^{1/2} R_k^\dagger P_\phi R_k \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle &\leq \sum_k \langle\phi| R_k \mathfrak{E}(P)^{1/2} R_k^\dagger P R_k \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle \\ &\leq \sum_k \langle\phi| R_k \mathfrak{E}(P)^{1/2} \left(\sum_l R_l^\dagger P R_l \right) \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle \end{aligned}$$

because we added only positive terms since $\mathfrak{R}_{op}$ is completely positive and $|\phi_i\rangle \langle\phi_i|$ is a positive map for every $1 \leq i \leq n$. Due to the fact that $\mathfrak{R}_{op}$ is trace preserving, i.e. $\sum_l R_l^\dagger P R_l = P_\mathfrak{C}$ it is

$$\sum_k \langle\phi| R_k \mathfrak{E}(P)^{1/2} \left(\sum_l R_l^\dagger P R_l \right) \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle = \sum_k \langle\phi| R_k \mathfrak{E}(P) R_k^\dagger |\phi\rangle = \langle\phi| \mathfrak{R}_{op} \circ \mathfrak{E}(P) |\phi\rangle .$$

Now, for any $1 \leq i \leq n$ we consider

$$\rho_i := \mathfrak{R}_{op} \circ \mathfrak{E}(|\phi_i\rangle \langle\phi_i|) = \sum_{1 \leq k,l \leq d} \alpha_{kl}^{(i)} |\phi_k\rangle \langle\phi_l|$$

for coefficients $\alpha_{kl}^{(i)} := \langle\phi_k| \rho_i |\phi_l\rangle$ satisfying the normalization condition $\sum_k \alpha_{kk}^{(i)} = 1$ on the one hand. On the other hand, from positivity of ρ_i for any i these coefficients also fulfill $\alpha_{kl}^{(i)} \geq 0$ for all k. The definition of the optimal fidelity loss (3.3) then directly gives us

$$\alpha_{ii}^{(i)} = \langle\phi_i| \rho_i |\phi_i\rangle = F^2 \left[|\phi_i\rangle, \mathfrak{R}_{op} \circ \mathfrak{E}(|\phi_i\rangle \langle\phi_i|) \right] \geq 1 - \eta_{op}.$$

In combination with the normalization condition this implies that

$$\eta_{op} \geq 1 - \alpha_{ii}^{(i)} = \sum_k \alpha_{kk}^{(i)} - \alpha_{ii}^{(i)} = \sum_{k \neq i} \alpha_{kk}^{(i)}.$$

This in combination with the positivity of ρ_i for any i, implies that $\alpha_{kk}^{(i)} \leq \eta_{op}$ for all $k \neq i$.

Now, we can put these bounds together to gain the upper bound:

$$\langle\phi| \mathfrak{R}_{op} \circ \mathfrak{E}(P) |\phi\rangle = \langle\phi_1| \sum_{i=1}^n \rho_i |\phi_1\rangle = \alpha_{11}^{(1)} + \sum_{i=2}^n \alpha_{11}^{(i)} \leq 1 + (n-1)\eta_{op}$$

where $\phi = \phi_1$ by definition. Remembering our starting point, this means

$$\sum_l \left| \langle\phi| E_l^\dagger \mathfrak{E}(P)^{-1/2} E_l |\phi\rangle \right|^2 \leq 1 + (n-1)\eta_{op}. \tag{21}$$

At last, we combine the upper bounds we found for each sum and conclude

$$\begin{aligned} F^2\left(|\phi\rangle, R_{op} \circ \mathfrak{E}\right) &\leq \sqrt{\sum_k \left|\langle\phi| R_k \mathfrak{E}(P)^{1/2} R_k^\dagger |\phi\rangle\right|^2 \sum_l \left|\langle\phi| E_l^\dagger \mathfrak{E}(P)^{-1/2} E_l |\phi\rangle\right|^2} \\ &\leq \sqrt{1+(n-1)\eta_{op}} \cdot F\left(|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}\right) \end{aligned}$$

□

From theorem (3.4) we easily deduce the following result for the fidelity loss of the transpose channel η_p.

(3.5) Corollary
Let a code $\mathfrak{C}$ affected by an error $\mathfrak{E}$ and its transpose channel $\mathfrak{R}_p$ be given. Then, the fidelity loss η_p satisfies

$$\eta_{op} \leq \eta_p \leq \eta_{op} f\left(\eta_{op}; n\right) \tag{22}$$

where $f(\eta; n)$ is given by

$$f(\eta; n) := \frac{n+1-\eta}{1+(n-1)\eta} = (n+1) + O(\eta). \tag{23}$$

◇

Proof
Since $\eta_{op} \leq \eta_p$ is true by definition of the optimal fidelity loss it is left to show the second inequality. For this, define

$$1 - \eta_{p,\phi} := F^2\left(|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}\right) \quad \text{for any } |\phi\rangle \in \mathfrak{C}.$$

Theorem (3.4) then yields

$$\begin{aligned} 1 - \eta_{op} &\leq F^2\left(|\phi\rangle, \mathfrak{R}_{op} \circ \mathfrak{E}\right) \leq \sqrt{1+(n-1)\eta_{op}}\sqrt{1-\eta_{p,\phi}} \\ \Rightarrow \frac{(1-\eta_{op})^2}{1+(n-1)\eta_{op}} &\leq 1 - \eta_{p,\phi} \\ \Rightarrow \eta_{p,\phi} &\leq \frac{\eta_{op}(n+1-\eta_{op})}{1+(n-1)\eta_{op}} = \eta_{op} f\left(\eta_{op}; n\right) \end{aligned}$$

for all $\eta_{p,\phi}$ and therefore also for η_p. □

The second inequality exactly shows that $\mathfrak{R}_p$ is only worse by the factor of $n+1$ than the optimal recovery. For the case of a low dimensional code this is a very good approximation. Indeed, for the case we consider - a code encoding a single qubit - the factor is only 3. On the other hand, as n gets large the function $\eta f(\eta, n)$ converges to 1 and (3.5) becomes trivial. Concluding, in case of perfect QEC it is $\eta_{op} = 0$ and (3.5) yields $\eta_p = 0$ as already expected from lemma (2.8).

3.3 AQEC Conditions

As in the case of perfect QEC we want to find conditions for a code $\mathfrak{C}$ to be approximately or rather, more general, ϵ-correctable. Fortunately, we will be able to upgrade the QEC conditions to AQEC conditions. The upgrade is done by adding an operator to the right-hand side of equation (12) for each i, j. It is obvious, however, that this interferences with the quality of possible recoveries of a code. In fact, we will see that it is directly related to the fidelity loss of the transpose channel.

(3.6) Lemma
Let P be the projector onto a quantum code $\mathfrak{C}$ affected by a TP error $\mathfrak{E} \sim \{E_i | 1 \leq i \leq m\}$ such that with a set $\Delta_{ij} \in O(\mathfrak{C})$ of traceless operators it is

$$PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P = \beta_{ij} P + \Delta_{ij} \quad \text{for all } 1 \leq i, j \leq m \tag{24}$$

where $\beta \in \mathbb{C}^{m \times m}$.
Then, the fidelity loss of the transpose channel η_p is given by

$$\eta_p := \max_{|\phi\rangle \in \mathfrak{C}} \sum_{1 \leq i,j \leq m} \left(\langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2 \right). \tag{25}$$

◇

Proof
As already encountered before in (14), the left-hand side of equation (33) is a Kraus operator of $\mathfrak{R}_p \circ \mathfrak{E}$. Due to the fact that $\mathfrak{E}$ and $\mathfrak{R}_p$ are both trace preserving on $\mathfrak{C}$, the composition $\mathfrak{R}_p \circ \mathfrak{E}$ is TP, too. These two details combined yield

$$\begin{aligned} P &= \sum_{ij} \left(PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P \right)^\dagger \left(PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P \right) \\ &= \sum_{ij} P \left[\left|\beta_{ij}\right|^2 + \Delta_{ij}^\dagger \Delta_{ij} + \beta_{ij}^* \Delta_{ij} + \beta_{ij} \Delta_{ij}^\dagger \right] P \end{aligned}$$

or equivalently

$$\sum_{ij} \beta_{ij}^* \Delta_{ij} + \beta_{ij} \Delta_{ij}^\dagger = 1 - \sum_{ij} \left[\left|\beta_{ij}\right|^2 + \Delta_{ij}^\dagger \Delta_{ij} \right] \tag{26}$$

on the support of $\mathfrak{C}$.
On the other hand, the fidelity of the transpose channel for any $|\phi\rangle \in \mathfrak{C}$ is given by (with $P_\phi := |\phi\rangle \langle\phi|$)

$$\begin{aligned} F^2 \left[|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}(P_\phi) \right] &= \langle\phi| \mathfrak{R}_p \circ \mathfrak{E}(P_\phi) |\phi\rangle = \langle\phi| \sum_{ij} (\beta_{ij} P + \Delta_{ij}) P_\phi (\beta_{ij}^* P + \Delta_{ij}^\dagger) |\phi\rangle \\ &= \sum_{ij} \underbrace{\left|\langle\phi| \beta_{ij} |\phi\rangle\right|^2}_{=\left|\beta_{ij}\right|^2} + \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2 + \langle\phi| \beta_{ij}^* \Delta_{ij} + \beta_{ij} \Delta_{ij}^\dagger |\phi\rangle. \end{aligned}$$

With equation (35) we can substitute the last summand and get

$$F^2\left[|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}(P_\phi)\right] = 1 - \sum_{ij} \langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2 \tag{27}$$

which gives the desired equation (25) for η_p since

$$\begin{aligned}\eta_p &= 1 - \min_{|\phi\rangle\in\mathfrak{C}} F^2 = 1 - \min_{|\phi\rangle\in\mathfrak{C}} \left[1 - \sum_{ij}\left(\langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2\right)\right] \\ &= \max_{|\phi\rangle\in\mathfrak{C}} \sum_{ij}\left(\langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2\right).\end{aligned}$$

As a conclusion, $\eta_p \in [0,1]$ since $F^2\left[|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}(P_\phi)\right] \in [0,1]$. □

Thus, for a given code $\mathfrak{C}$ the fidelity loss of the transpose channel only depends on the appendages Δ_{ij}. Together with corollary (3.5) one even can evaluate the fidelity of the optimal recovery for the case of lemma (3.6). In other words, we are now able to upgrade the QEC conditions (2.9) by combining the knowledge from lemma (3.6) and corollary (3.5).

(3.7) Theorem (AQEC Conditions, Ng and Mandayam)
Let P be the projector onto a quantum code $\mathfrak{C}$ of dimension n affected by an TP-error $\mathfrak{E} \sim \{E_i | 1 \leq i \leq m\}$ such that, with a set $\Delta_{ij} \in O(\mathfrak{C})$ of traceless operators, it is

$$PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P = \beta_{ij} P + \Delta_{ij} \quad \text{for all } 1 \leq i,j \leq m$$

where $\beta \in \mathbb{C}^{m\times m}$.
Then, for every $\epsilon \in [0,1]$ it exists $\eta_p \in [0,1]$ (the fidelity loss in using the transpose channel $\mathfrak{R}_p$ as the recovery map) given by (see lemma 3.6)

$$\eta_p = \max_{|\phi\rangle\in\mathfrak{C}} \sum_{1\leq i,j\leq m} \left(\langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2\right)$$

such that

1. $\mathfrak{C}$ is ϵ-correctable if $\eta_p \leq \epsilon$
2. $\mathfrak{C}$ is ϵ-correctable only if $\eta_p \leq \epsilon f(\epsilon; n)$ for

   $$f(\epsilon; n) = \frac{(n+1) - \epsilon}{1 + (n-1)\epsilon}$$

   as defined in equation (23). ◇

In the case of the absence of any interference denoted by the Δ_{ij}, i. e. $\Delta_{ij} = 0$ for all i and j, we have perfect QEC and this theorem transforms into theorem (2.9).

Proof
We already proved in lemma (3.6) that the fidelity loss η_p fulfills the claimed equation. The two conditions 1. and 2. also follow directly from corollary (3.5). □

The fidelity loss of the transpose channel η_p is not yet clearly visualizable. To change this, consider the projection map $\mathfrak{P}_\phi^\perp := P_\phi^\perp(\cdot)P_\phi^\perp$ with $P_\phi^\perp := P - P_\phi$ being the projector onto the orthogonal complement of $|\phi\rangle\langle\phi|$ in $\mathfrak{C}$. Together with the completely positive map Δ induced by the Kraus representation $\{\Delta_{ij}\}$ this yields

(3.8) Remark
Let P be the projector onto a quantum code $\mathfrak{C}$ and let (33) be fulfilled for $\Delta \sim \{\Delta_{ij}\}$. Then, the fidelity loss η_p can be written as

$$\eta_p = \max_{|\phi\rangle\in\mathfrak{C}} tr\left[\mathfrak{P}_\phi^\perp \circ \Delta(P_\phi)\right]. \tag{28}$$

◇

Written in this form, it is clear how the fidelity loss η_p arises from the interference of the Δ_{ij}. Indeed, the amount of fidelity missing is exactly the same as the orthogonal amount (in respect to $|\phi\rangle\langle\phi|$) of the image of Δ applied on $|\phi\rangle\langle\phi|$.

Proof
We proof this equation by deducing it to the known equation (25) of η_p.
Let $P = \sum_i |\phi_i\rangle\langle\phi_i|$. We observer that

$$\mathfrak{P}_\phi^\perp \circ \Delta\left(P_\phi\right) = (P - P_\phi)\Delta(P - P_\phi) = \underbrace{P\Delta P}_{=:A} - \underbrace{P\Delta P_\phi}_{=:B} - \underbrace{P_\phi\Delta P}_{=B^\dagger} + \underbrace{P_\phi\Delta P_\phi}_{=:C} \tag{29}$$

where we used the shorthand $\Delta := \Delta\left(P_\phi\right)$. Then, it is

$$\begin{aligned} tr\,[A] &= \sum_i \langle\phi_i|\,\Delta\,|\phi_i\rangle = \sum_i\sum_{kl} \langle\phi_i|\,\Delta_{kl}\,|\phi\rangle\langle\phi|\,\Delta_{kl}^\dagger\,|\phi_i\rangle = \sum_{kl}\langle\phi|\,\Delta_{kl}^\dagger(\sum_i|\phi_i\rangle\langle\phi_i|)\Delta_{kl}\,|\phi\rangle \\ &= \sum_{kl}\langle\phi|\,\Delta_{kl}^\dagger\Delta_{kl}\,|\phi\rangle \end{aligned}$$

as well as

$$\begin{aligned} tr\,[C] &= \sum_i \langle\phi_i|\phi\rangle\langle\phi|\,\Delta\,|\phi\rangle\langle\phi|\phi_i\rangle = \langle\phi|\,\Delta\,|\phi\rangle \\ &= \sum_{kl} |\langle\phi|\,\Delta_{kl}\,|\phi\rangle|^2 \end{aligned}$$

while the last equality follows from (3.2.2). We continue by rearranging the sums:

$$\begin{aligned} tr\left[\mathfrak{P}_\phi^\perp \circ \Delta(P_\phi)\right] &\overset{\text{eq. (29)}}{=} tr\left[A - B - B^\dagger + D\right] = tr\left[A - D - (B - D) - (B^\dagger - D)\right] \\ &= tr\,[A - D] - tr\,[B - D] - tr[\underbrace{B^\dagger - D}_{=(B-D)^\dagger}]. \end{aligned} \tag{30}$$

As a next step, we will show that the last two traces are equal to zero. Due to the fact that they are the complex conjugate of each other, it is sufficient to show this for one of them. For this, consider

$$tr\left[B-D\right] = tr\left[P\Delta P_\phi - P_\phi \Delta P_\phi\right] = tr\left[(P-P_\phi)\Delta P_\phi\right] = tr\left[P_\phi^\perp \Delta P_\phi\right] = 0 \tag{31}$$

where the trace is zero due to the fact that $P_\phi^\perp$ and P_ϕ have disjunct supports. Hence, we have

$$\begin{aligned} tr\left[\mathfrak{P}_\phi^\perp \circ \Delta(P_\phi)\right] &\overset{\text{eq. (30)}}{=} tr\left[A-D\right] - tr\left[B-D\right] - tr\left[B-D\right]^\dagger \\ &\overset{\text{eq. (31)}}{=} \sum_{kl} \langle\phi|\, \Delta_{kl}^\dagger \Delta_{kl}\, |\phi\rangle - \sum_{kl} |\langle\phi|\, \Delta_{kl}\, |\phi\rangle|^2 \end{aligned}$$

which is exactly equation (25). □

The purpose of the AQEC conditions is the same as of the perfect QEC conditions. It provides a way to check for any ϵ whether a given code $\mathfrak{C}$ is good enough to be ϵ-correctable - without the knowledge of the optimal recovery!
Thus, having a code $\mathfrak{C}$ of dimension n, lemma (3.6) shows us how to compute the fidelity loss η_p with knowing the code $\mathfrak{C}$ and the affecting error $\mathfrak{E}$ only. Then, if $\eta_p \leq \epsilon$ the code is good enough for use. On the other hand, if $\eta_p > \epsilon f(\epsilon, n)$ (i.e. the code violates the second condition), the code does not match our requirements and can be dropped. One could ask how to proceed if the code violates the first condition but fulfills the second, i.e. $\epsilon < \eta_p \leq \epsilon f(\epsilon, n)$. In this case, we have no justified information on whether or not the code is useful. The code could fulfill $\eta < \epsilon$ for a particular recovery operation, of course. However, the opposite could also be the case. Fortunately, this gap depends on the dimension n of the code and therefore is quite small for small n. Even though it could be possible to further minimize the gap by finding another recovery map with corresponding fidelity loss than the transpose channel, it is unlikely that the gap vanishes completely [4, p. 8].
We are theoretically able to compute the fidelity loss η_p for any Code $\mathfrak{C}$. The realisation, however, could be very difficult to accomplish since it involves a maximization over all states in the code space. Luckily, there is a much quicker way to check the usability of a code if we relax the first condition of the AQEC conditions (3.7) a bit.

(3.9) Corollary
In terms of (3.7), $\mathfrak{C}$ is ϵ-correctable for a given $\epsilon \in [0, 1]$ if

$$\|\Delta_{sum}\|_{op} \leq \epsilon,$$

with $\Delta_{sum} := \sum_{ij} \Delta_{ij}^\dagger \Delta_{ij}$ ◇

Here, $\|\cdot\|_{op}$ denotes the operator norm. Due to the fact that, by definition, Δ_{sum} is a positive semidefinite operator its operator norm takes a simple form. In particular, the norm is given by the maximum eigenvalue of Δ_{sum} which is generally easy to compute.

Proof
Let $\|\Delta_{sum}\|_{op} \leq \epsilon$ for some $\epsilon \in [0,1]$. It is obvious that

$$\sum_{ij} \left(\langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2 \right) \leq \sum_{ij} \langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle = \langle\phi| \Delta_{sum} |\phi\rangle .$$

On the other hand, it is directly followed from the definition of the operator norm that

$$\|\Delta_{sum}\| = \max_{|\phi\rangle \in \mathfrak{C}} \langle\phi| \Delta_{sum} |\phi\rangle .$$

If we combine these relations, we have $\eta_p \leq \|\Delta_{sum}\| \leq \epsilon$. Therefore, the condition 1 of the AQEC conditions (3.7) is fulfilled such that $\mathfrak{C}$ is ϵ-correctable. □

3.4 AQEC for non trace preserving errors

As defined in (1.5), a quantum operation does not need to be trace preserving. We want to examine the effects on AQEC conditions (3.7) if this requirement on the error is dropped.
In particular, in terms of theorem (3.7) let $\mathfrak{E}$ be a non trace preserving operation. Then, it is $\sum_i E_i^\dagger E_i \leq I$ for the Kraus operators of $\mathfrak{E}$. We will especially be interested in the case that $\mathfrak{E}$ is nearly trace preserving. For this, we state that

$$\sum_i E_i^\dagger E_i = I - \delta B \tag{32}$$

for $\delta \in \mathbb{R}$ and a positive operator B. The trace preserving condition on the error occurred at lemma (3.6) and was inherited by the AQEC conditions theorem (3.7). Fortunately, we can generalize lemma (3.6) to work with errors like (32).

(3.10) Lemma
Let P be the projector onto a quantum code $\mathfrak{C}$ affected by an error $\mathfrak{E} \sim \{E_i | 1 \leq i \leq m\}$ with $\sum_i E_i^\dagger E_i = I - \delta B$ for $\delta \in \mathbb{R}$ and positive B. Assume that, with a set $\Delta_{ij} \in O(\mathfrak{C})$ of traceless operators, it is

$$PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P = \beta_{ij} P + \Delta_{ij} \quad \text{for all } 1 \leq i,j \leq m \tag{33}$$

where $\beta \in \mathbb{C}^{m \times m}$.
Then, the fidelity loss of the transpose channel η_p is given by

$$\eta_p := \max_{|\phi\rangle \in \mathfrak{C}} \sum_{1 \leq i,j \leq m} \left(\langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2 \right) + \delta \cdot \langle\phi| B |\phi\rangle . \tag{34}$$

◇

Proof
The proof differs only in a few details induced by the change on the error condition from the proof of lemma (3.6):

As already encountered before in (14), the left-hand side of equation (33) is a Kraus operator of $\mathfrak{R}_p \circ \mathfrak{E}$. Due to the fact that $\mathfrak{R}_p$ is trace preserving on $\mathfrak{C}$ and $\mathfrak{E}$ fulfills $\sum_i E_i^\dagger E_i = I - \delta B$ for some $\delta \in \mathbb{R}$ and positive operator B, it is for the Kraus operators of $\mathfrak{R}_p \circ \mathfrak{E}$:

$$\begin{aligned} P(I-\delta B)P &= \sum_{ij} \left(PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P \right)^\dagger \left(PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P \right) \\ &= \sum_{ij} P \left[\left|\beta_{ij}\right|^2 + \Delta_{ij}^\dagger \Delta_{ij} + \beta_{ij}^* \Delta_{ij} + \beta_{ij} \Delta_{ij}^\dagger \right] P \end{aligned}$$

such that

$$\sum_{ij} \beta_{ij}^* \Delta_{ij} + \beta_{ij} \Delta_{ij}^\dagger = (1-\delta B) - \sum_{ij} \left[\left|\beta_{ij}\right|^2 + \Delta_{ij}^\dagger \Delta_{ij} \right] \tag{35}$$

on the support of $\mathfrak{C}$.

On the other hand, the fidelity of the transpose channel for any $|\phi\rangle \in \mathfrak{C}$ is given by (with $P_\phi := |\phi\rangle \langle \phi|$)

$$\begin{aligned} F^2 \left[|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}(P_\phi) \right] &= \langle \phi | \mathfrak{R}_p \circ \mathfrak{E}(P_\phi) |\phi\rangle = \langle \phi | \sum_{ij} (\beta_{ij} P + \Delta_{ij}) P_\phi (\beta_{ij}^* P + \Delta_{ij}^\dagger) |\phi\rangle \\ &= \sum_{ij} \underbrace{\left| \langle \phi | \beta_{ij} |\phi\rangle \right|^2}_{=\left|\beta_{ij}\right|^2} + \left| \langle \phi | \Delta_{ij} |\phi\rangle \right|^2 + \langle \phi | \beta_{ij}^* \Delta_{ij} + \beta_{ij} \Delta_{ij}^\dagger |\phi\rangle . \end{aligned}$$

With equation (35) we can substitute the last addend and get

$$F^2 \left[|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}(P_\phi) \right] = 1 - \delta \langle \phi | B |\phi\rangle - \sum_{ij} \langle \phi | \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left| \langle \phi | \Delta_{ij} |\phi\rangle \right|^2$$

which gives the desired equation (25) for η_p since

$$\begin{aligned} \eta_p &= 1 - \min_{|\phi\rangle \in \mathfrak{C}} F^2 = 1 - \min_{|\phi\rangle \in \mathfrak{C}} \left[1 - \delta \langle \phi | B |\phi\rangle - \sum_{ij} \left(\langle \phi | \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left| \langle \phi | \Delta_{ij} |\phi\rangle \right|^2 \right) \right] \\ &= \max_{|\phi\rangle \in \mathfrak{C}} \delta \cdot \langle \phi | B |\phi\rangle + \sum_{ij} \left(\langle \phi | \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left| \langle \phi | \Delta_{ij} |\phi\rangle \right|^2 \right) . \end{aligned}$$

As a conclusion, $\eta_p \in [0,1]$ since $F^2 \left[|\phi\rangle, \mathfrak{R}_p \circ \mathfrak{E}(P_\phi) \right] \in [0,1]$. □

The generalization of the AQEC conditions from NG and Mandayam to non trace preserving errors can now be concluded.

(3.11) Theorem (AQEC Conditions, generalized)
Let P be the projector onto a quantum code $\mathfrak{C}$ of dimension n affected by an error $\mathfrak{E} \sim \{E_i | 1 \leq i \leq m\}$ with $\sum_i E_i^\dagger E_i = I - \delta B$ for $\delta \in \mathbb{R}$ and positive B. Assume that, with a set $\Delta_{ij} \in O(\mathfrak{C})$ of traceless operators, it is

$$PE_i^\dagger \mathfrak{E}(P)^{-1/2} E_j P = \beta_{ij} P + \Delta_{ij} \quad \text{for all } 1 \leq i, j \leq m$$

where $\beta \in \mathbb{C}^{m \times m}$.
Then, for every $\epsilon \in [0, 1]$ it exists $\eta_p \in [0, 1]$ (the fidelity loss in using the transpose channel $\mathfrak{R}_p$ as the recovery map) given by (see lemma 3.10)

$$\eta_p = \max_{|\phi\rangle \in \mathfrak{C}} \delta \cdot \langle\phi| B |\phi\rangle + \sum_{1 \leq i,j \leq m} \left(\langle\phi| \Delta_{ij}^\dagger \Delta_{ij} |\phi\rangle - \left|\langle\phi| \Delta_{ij} |\phi\rangle\right|^2 \right)$$

such that

1. $\mathfrak{C}$ is ϵ-correctable if $\eta_p \leq \epsilon$
2. $\mathfrak{C}$ is ϵ-correctable only if $\eta_p \leq \epsilon f(\epsilon; n)$ for

$$f(\epsilon; n) = \frac{(n+1) - \epsilon}{1 + (n-1)\epsilon}$$

as defined in equation (23). ◇

So, we see that for $\delta \ll \epsilon$ the effect of a non trace preserving error can be practically neglected.

4 π Cat State Code and AQEC

After having the AQEC conditions we want to apply them onto a single qubit quantum code , the π cat state code, to evaluate its capability. In spite of the fact that the occurring error is not trace preserving, our examinations will not be bothered due to the generalization of the AQEC conditions (3.11) we have just accomplished.

4.1 π Cat State Code, Introduction

The code we want to examine is based on a single bosonic mode with creation and annihilation operators a resp. $a^\dagger$. In particular, a qubit is encoded into the states

$$|\bar{0}_+\rangle := \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{N_+}}, \quad |\bar{1}_+\rangle := \frac{|i\alpha\rangle + |-i\alpha\rangle}{\sqrt{N_+}} \tag{36}$$

where $|\alpha\rangle$ is a coherent state and $N_\pm$ is a normalization factor, i.e.

$$\alpha := e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad N_\pm := 2 \cdot (1 \pm exp(-2|\alpha|^2)) \tag{37}$$

Due to the fact, that the multiplication with i means a rotation of the argument α by the angle π in the complex plane this code is also called *π-cat state code*. It was shown that this code approximately protects against photon loss [3]. The damping process can be modeled with the Lindblad equation:

$$\dot{\rho} = -i\,[H, \rho] + \mathfrak{L}(\rho)$$

where H is the usual Hamiltonian which generates the unitary evolution and $\mathfrak{L}(\rho)$ is the so called *Lindbladian* which generates the non-unitary evolution. The Lindbladian can be derived from an infinitesimal evolution described by Kraus operators L_k (called *quantum-jump* or Lindblad operators) such that

$$\mathfrak{L}(\rho) = \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right). \tag{38}$$

If we take short times τ we get $\rho(\tau) = E_0 \rho E_0^\dagger + \sum_k E_k \rho E_k^\dagger$ with (see [6, p. 10]):

$$E_0 \approx I - i\tau H - \frac{1}{2}\tau \sum_k L_k^\dagger L_k = I - O(\tau) \text{ and } E_k \approx \sqrt{\tau} L_k.$$

In the case of the single bosonic mode, this means

$$E_0 \approx e^{-\frac{\kappa\tau}{2} a^\dagger a} = e^{-\frac{\kappa\tau}{2} n}, \quad E_1 \approx \sqrt{\kappa\tau}\, a \tag{39}$$

where $\kappa \in \mathbb{R}_+$ is a damping constant. We have a trace preserving condition

$$\sum_i E_i^\dagger E_i = I + O((\kappa\tau a^\dagger a)^2)$$

to quadratic terms of $\kappa\tau n$.
Since $a|\alpha\rangle = \alpha|\alpha\rangle$, we have

$$E_1|\bar{0}_+\rangle = \sqrt{\kappa\tau}\alpha\frac{|\alpha\rangle - |-\alpha\rangle}{\sqrt{N_+}} = \alpha\sqrt{\kappa\tau\frac{N_-}{N_+}}|\bar{0}_-\rangle$$

and similar

$$E_1|\bar{1}_+\rangle = \sqrt{\kappa\tau}\alpha\frac{|i\alpha\rangle - |-i\alpha\rangle}{\sqrt{N_+}} = i\alpha\sqrt{\kappa\tau\frac{N_-}{N_+}}|\bar{1}_-\rangle$$

with N_- already defined and

$$|\bar{0}_-\rangle := \frac{|\alpha\rangle - |-\alpha\rangle}{\sqrt{N_-}}, \quad |\bar{1}_-\rangle := \frac{|i\alpha\rangle - |-i\alpha\rangle}{\sqrt{N_-}}.$$

On the other hand, it is

$$\begin{aligned} e^{-\frac{\kappa\tau}{2}n}|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{(\alpha\cdot e^{-\frac{\kappa\tau}{2}})^n}{\sqrt{n!}}|n\rangle \\ &= e^{-\frac{|\alpha|^2}{2}}exp\left(-\frac{|\alpha\cdot e^{-\frac{\kappa\tau}{2}}|^2}{2}\right)\left|\underbrace{\alpha e^{-\frac{\kappa\tau}{2}}}_{=:\alpha_{\kappa\tau}}\right\rangle \\ &= exp\left(-\frac{|\alpha|^2}{2}(1-e^{-\kappa\tau})\right)|\alpha_{\kappa\tau}\rangle \end{aligned}$$

such that

$$\langle\bar{0}_+|E_0^\dagger E_0|\bar{0}_+\rangle = \frac{cosh(|\alpha_{\kappa\tau}|^2)}{cosh(|\alpha|^2)} = \langle\bar{1}_+|E_0^\dagger E_0|\bar{1}_+\rangle. \tag{40}$$

as we will show later. Therefore, the image $E_0(\mathfrak{C})$ is spanned by

$$|\bar{0}_{\kappa\tau}\rangle := \sqrt{\frac{cosh(|\alpha|^2)}{cosh(|\alpha_{\kappa\tau}|^2)}}\cdot E_0|\bar{0}_+\rangle, \quad |\bar{1}_{\kappa\tau}\rangle := \sqrt{\frac{cosh(|\alpha|^2)}{cosh(|\alpha_{\kappa\tau}|^2)}}\cdot E_0|\bar{1}_+\rangle \tag{41}$$

which are both normalized.

4.2 π Cat State Code, Analysis

To continue our examination of the π-cat state code it is urgent to know the relations between the different states and subspaces. We start with a formula for the bracket of two coherent states.

(4.1) Proposition
Let $|\alpha\rangle$ and $|\beta\rangle$ be two arbitrary coherent states as defined in (37). Then, it is

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\beta^*\alpha)}. \qquad \diamond$$

Proof
Simple calculations yield:

$$\begin{aligned}\langle\alpha|\beta\rangle &= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\sum_{n,m}\frac{\alpha^n\beta^{*m}}{\sqrt{n!}\sqrt{m!}}\langle n|m\rangle \\ &= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\sum_{n}\frac{(\alpha\beta^*)^n}{n!} = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\beta^*\alpha)} \qquad \square\end{aligned}$$

Due to the fact, that the coherent states we deal with are of the form $|k\alpha\rangle$ with some $k\in\mathbb{C}, |k|=1$ the formula (4.1) simplifies into

(4.2) Corollary
Let $k,l\in\mathbb{C}$ with $|k|=|l|=1$. Then, it is

$$\langle l\alpha|k\alpha\rangle = e^{-|\alpha|^2(1-l^*k)} = \langle l^*\alpha|k\alpha\rangle$$

and especially

1. $\langle k\alpha|l\alpha\rangle = \langle -k\alpha|-l\alpha\rangle$
2. $\langle -k\alpha|l\alpha\rangle = \langle k\alpha|-l\alpha\rangle$ $\diamond$

We will use this corollary frequently in the following, starting with the proof of (40): With (4.2) we have

$$\begin{aligned}\langle\bar{0}_+|\,E_0^\dagger E_0\,|\bar{0}_+\rangle &= \frac{1}{N_+}exp(-|\alpha|^2(1-e^{\kappa\tau}))\,* \\ &* \; (\underbrace{\langle\alpha_{\kappa\tau}|\alpha_{\kappa\tau}\rangle}_{=1}+\underbrace{\langle-\alpha_{\kappa\tau}|\alpha_{\kappa\tau}\rangle+\langle\alpha_{\kappa\tau}|-\alpha_{\kappa\tau}\rangle}_{=2exp(-2|\alpha_{\kappa\tau}|^2)\ (4.2)}+\underbrace{\langle-\alpha_{\kappa\tau}|-\alpha_{\kappa\tau}\rangle}_{=1}) \\ &= \frac{2e^{-|\alpha|^2}}{N_+}exp(\underbrace{|\alpha|^2e^{\kappa\tau}}_{=|\alpha_{\kappa\tau}|^2})(1+e^{-2|\alpha_{\kappa\tau}|^2}) = \frac{2e^{-|\alpha|^2}}{2(1+e^{-2|\alpha|^2})}(e^{|\alpha_{\kappa\tau}|^2}+e^{-|\alpha_{\kappa\tau}|^2}) \\ &= \frac{cosh(|\alpha_{\kappa\tau}|^2)}{cosh(|\alpha|^2)} = \langle\bar{1}_+|\,E_0^\dagger E_0\,|\bar{1}_+\rangle\,.\end{aligned}$$

while the last equation results if we substitute α with $i\alpha$.
The code space $\mathfrak{C}$ is spanned by the two states $|\bar{0}_+\rangle$ and $|\bar{1}_+\rangle$. However, by applying the bracket formula (4.2) we encounter

$$\begin{aligned}\langle\bar{0}_+|\bar{1}_+\rangle &= \frac{1}{N_+}(\langle\alpha|i\alpha\rangle+\langle\alpha|-i\alpha\rangle+\langle-\alpha|i\alpha\rangle+\langle-\alpha|-i\alpha\rangle) \\ &= \frac{1}{N_+}(2\,\langle\alpha|i\alpha\rangle+2\,\langle\alpha|-i\alpha\rangle) = \frac{2}{N_+}(e^{-|\alpha|^2(1+i)}+e^{-|\alpha|^2(1-i)}) \\ &= \frac{4\cdot e^{-|\alpha|^2}}{N_+}cos(|\alpha|^2) = \frac{cos(|\alpha|^2)}{cosh(|\alpha|^2)}.\end{aligned}$$

So, for the majority of α the states $|\bar{0}_+\rangle$ and $|\bar{1}_+\rangle$ are not orthogonal to each other. Since we frequently use the orthogonal projector onto the code space $\mathfrak{C}$ in the calculations, it is useful to find an orthonormal basis for $\mathfrak{C}$ with the Gram-Schmidt process, i.e.

(4.3) Proposition
The orthogonal projector ,denoted P_c, onto the code space $\mathfrak{C}$ is given by $P_c = |\bar{0}_+\rangle\langle\bar{0}_+| + |\tilde{1}_+\rangle\langle\tilde{1}_+|$ with

$$|\tilde{1}_+\rangle := (|\bar{0}_+\rangle - \frac{cos(|\alpha|^2)}{cosh(|\alpha|^2)}|\bar{1}_+\rangle)/\sqrt{1-\left(\frac{cos(|\alpha|^2)}{cosh(|\alpha|^2)}\right)^2}. \qquad \diamond$$

The error operations E_i each maps the code space $\mathfrak{C}$ onto its corresponding error space. Any recovery operator then has to map each error space back onto the code space. If the error spaces are somehow related to each other the calculation of the recovery operators can get very complicated. Fortunately, in the case of the π-cat state code it is not that complicated due to the fact that the two error spaces are orthogonal to each other. This can be easily seen by:

(4.4) Proposition
For $\alpha, \beta \in \mathbb{C}$ the coherent states $|\pm\beta\rangle$ and $|\pm\alpha\rangle$ fulfill

$$(\langle\beta| + \langle-\beta|)(|\alpha\rangle - |-\alpha\rangle) = 0 \qquad \diamond$$

Proof
Applying (4.2) directly yields the result. □

Now, on the one hand we have the error space that corresponds to E_1 spanned by $|\bar{0}_-\rangle$ and $|\bar{1}_-\rangle$ which are both of the form $(|\beta\rangle - |-\beta\rangle)$. On the other hand, we have $E_0(\mathfrak{C})$ spanned by $|\tilde{1}_+\rangle$ and $|\tilde{0}_+\rangle$ which are both of the form $(\langle\alpha| + \langle-\alpha|)$. Therefore, we know from (4.4) that $E_1(\mathfrak{C})$ and $E_0(\mathfrak{C})$ are orthogonal to each other. This means we can handle each independently in our calculations, especially when it comes to find the inverse of $\mathfrak{E}$ on its support (which we need in the transpose channel (2.8).
Before we can start with the calculation of the transpose channel and on the conditions of AQEC, we will find a base for each error space via Gram-Schmidt process and put all our analysis together in one final lemma.

(4.5) Lemma
Let $\mathfrak{C}$ be the code space spanned by $|\bar{0}_+\rangle$ and $|\bar{1}_+\rangle$ as defined in (36) which is affected by the error $\mathfrak{E}\tilde{}\{E_1, E_0\}$ as defined in (39). Then, the orthogonal projector onto the code space is given by

$$P_c = |\bar{0}_+\rangle\langle\bar{0}_+| + |\tilde{1}_+\rangle\langle\tilde{1}_+|$$

while the orthogonal projectors onto the two orthogonal error spaces $E_1(\mathfrak{C})$ and $E_0(\mathfrak{C})$ are given by

$$P_{E_1} = |\bar{0}_-\rangle\langle\bar{0}_-| + |\tilde{1}_-\rangle\langle\tilde{1}_-|$$

respectively

$$P_{E_0} = |\bar{0}_{\kappa\tau}\rangle \langle\bar{0}_{\kappa\tau}| + |\tilde{1}_{\kappa\tau}\rangle \langle\tilde{1}_{\kappa\tau}|.$$

Here, the tilde states are defined as followed:

$$\begin{aligned}
|\tilde{1}_+\rangle &= \frac{|\bar{1}_+\rangle - \frac{cos(|\alpha|^2)}{cosh(|\alpha|^2)}|\bar{0}_+\rangle}{\sqrt{1-\left(\frac{cos(|\alpha|^2)}{cosh(|\alpha|^2)}\right)^2}}, \\
|\tilde{1}_-\rangle &= \frac{|\bar{1}_-\rangle - i\frac{sin(|\alpha|^2)}{sinh(|\alpha|^2)}|\bar{0}_-\rangle}{\sqrt{1-\left(\frac{sin(|\alpha|^2)}{sinh(|\alpha|^2)}\right)^2}}, \\
|\tilde{1}_{\kappa\tau}\rangle &= \frac{|\bar{1}_{\kappa\tau}\rangle - \frac{cos(|\alpha_{\kappa\tau}|^2)}{cosh(|\alpha_{\kappa\tau}|^2)}|\bar{0}_{\kappa\tau}\rangle}{\sqrt{1-\left(\frac{cos(|\alpha_{\kappa\tau}|^2)}{cosh(|\alpha_{\kappa\tau}|^2)}\right)^2}}
\end{aligned}$$

◇

Proof

We have already shown most of the statements. In fact, the last two equations are the only new. They are direct results of the Gram-Schmidt process if

$$\langle\bar{0}_-|\bar{1}_-\rangle = i\frac{sin(|\alpha|^2)}{sinh(|\alpha|^2)}, \quad \langle\bar{0}_{\kappa\tau}|\bar{1}_{\kappa\tau}\rangle = \frac{cos(|\alpha_{\kappa\tau}|^2)}{cosh(|\alpha_{\kappa\tau}|^2)}. \tag{42}$$

However, with the knowledge of (4.2) we have

$$\begin{aligned}
\langle\bar{0}_-|\bar{1}_-\rangle &= \frac{1}{N_-}(\langle\alpha|i\alpha\rangle - \langle\alpha|-i\alpha\rangle - \langle-\alpha|i\alpha\rangle + \langle-\alpha|-i\alpha\rangle) \\
&= \frac{1}{N_-}(2\langle\alpha|i\alpha\rangle - 2\langle\alpha|-i\alpha\rangle) = \frac{2}{N_-}(e^{-|\alpha|^2(1+i)} - e^{-|\alpha|^2(1-i)}) \\
&= \frac{4\cdot e^{-|\alpha|^2}}{N_-} i sin(|\alpha|^2) = i\frac{sin(|\alpha|^2)}{sinh(|\alpha|^2)}.
\end{aligned}$$

and also

$$\begin{aligned}
\langle\bar{0}_{\kappa\tau}|\bar{1}_{\kappa\tau}\rangle &= \frac{cosh\left(|\alpha|^2\right)}{cosh\left(|\alpha_{\kappa\tau}|^2\right)}\langle\bar{0}_+|e^{-\kappa\tau n}|\bar{1}_+\rangle \\
&= \frac{cosh\left(|\alpha|^2\right)}{N_+}\cdot\frac{exp(|\alpha_{\kappa\tau}|^2 - |\alpha|^2)}{cosh\left(|\alpha_{\kappa\tau}|^2\right)} * \\
&* \ (\langle\alpha_{\kappa\tau}|i\alpha_{\kappa\tau}\rangle + \langle\alpha_{\kappa\tau}|-i\alpha_{\kappa\tau}\rangle + \langle-\alpha_{\kappa\tau}|i\alpha_{\kappa\tau}\rangle + \langle-\alpha_{\kappa\tau}|-i\alpha_{\kappa\tau}\rangle) \\
&= \frac{4e^{-|\alpha|^2}cosh\left(|\alpha|^2\right)}{N_+}\frac{e^{|\alpha_{\kappa\tau}|^2}}{cosh\left(|\alpha_{\kappa\tau}|^2\right)}\frac{e^{(i-1)|\alpha_{\kappa\tau}|^2} + e^{(-i-1)|\alpha_{\kappa\tau}|^2}}{2} \\
&= 1\cdot\frac{e^{i|\alpha_{\kappa\tau}|^2} + e^{-i|\alpha_{\kappa\tau}|^2}}{2cosh\left(|\alpha_{\kappa\tau}|^2\right)} = \frac{cos\left(|\alpha_{\kappa\tau}|^2\right)}{cosh\left(|\alpha_{\kappa\tau}|^2\right)}.
\end{aligned}$$

So, we have shown the equations (42) and the proof is completed. □

4.3 π Cat State Code and AQEC

Our next step will be the examination of the behavior of the upper bound of the fidelity loss given by $|\Delta_{sum}|$ as defined in (3.9). With our recent analysis on the π cat state code we are able to do all the calculations we need for this task. Surprisingly, at the end the result will be of the form

$$\Delta_{sum} = d_1(\kappa\tau, \alpha) \left|\bar{0}_+\right\rangle \left\langle\bar{0}_+\right| + d_2(\kappa\tau, \alpha) \left|\tilde{1}_+\right\rangle \left\langle\tilde{1}_+\right|$$

even though d_i are complicated real functions of $\kappa\tau$ and α.
If we fix $\kappa\tau \cdot n$ and plot the logarithm of the greatest eigenvalue of Δ_{sum} over values of the photon number n we see a fine linear decay over large ranges of n:

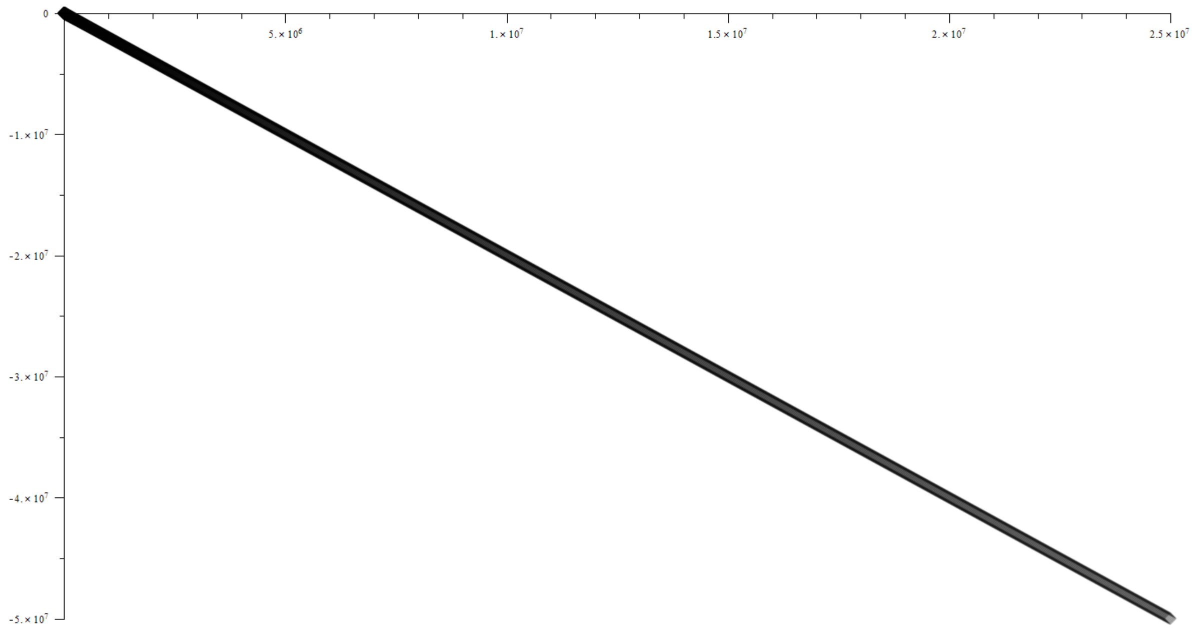

Figure 1: Logarithm of the greatest EV of Δ_{sum} for $\kappa\tau = 10^{-6}$ and number of photons n (x-axis).

Note: We can get the same figure for a variety of small values of $\kappa\tau \cdot n$, i.e. the slope is -2 for all tested values of $\kappa\tau \cdot n$.
This means, that we have at least a Gaussian decay of the fidelity loss given by

(4.6) Remark

$$\eta \leq |\Delta_{sum}| \approx e^{-2n}. \tag{43}$$

◇

This discovery gives us a really valuable upper bound for the fidelity loss due to its rapid decay. Although, the error in our example is not *completely* trace preserving but to terms of $(\kappa\tau \cdot n)^2$. We can take $\kappa\tau$ small enough to ensure that the fidelity loss is practically limited by this upper bound (43) due to (3.11).

5 Outlook

The limitations of the perfect quantum error correction conditions as defined in (2.5) are obvious. The code has to correct the error perfectly. As obvious therefore is the need of a less restrictive but still nearly as powerful QEC conditions. Hui Khoon Ng and Prabha Mandayam meet that need by using the transpose channel as general recovery operation to generalize the perfect QEC conditions to approximately quantum error correction conditions as defined in (3.7). This approach is admirable simple regarding the complexity of the problem. Further, it allows to use an algorithm to quickly test an approximately correction code of its maximal fidelity loss. Depending on the dimension of the code we can even give an explicit red or green light for any desired fidelity limit. As a next step, we generalized the AQEC conditions for non trace preserving quantum errors in (3.11). So, we ware able to testify the resistance of a large variety of combinations of quantum codes and errors.
However, this does not mean we are done yet. One possible topic is the definition of (A)QEC conditions for infinite-dimensional codes. Indeed, the formalism in this bachelor thesis fails for infinite-dimensional code spaces. Fortunately, Bény, Kempf and Kribs have already successfully adapted QEC conditions for infinite Hilbert spaces [1]. The adaption of AQEC conditions to this setting, however, has not been done yet.

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Erklärung

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